B.Sc. Part II vector calculus ($4^{th}$ Paper)

Q. A vector function $\vec{f}$ is the product of a scalar function and the gradient of a scalar function. Prove that

$$\vec{f} \cdot \text{curl}\,\vec{f} = 0.$$

Soln. Let the scalar function $= \psi$
and another scalar function $= \phi$

Given that $\vec{f} = \psi \text{ grad } \phi$

$$\Rightarrow \vec{f} = \psi \nabla \phi = \psi \left[ \vec{i} \frac{\partial}{\partial x} + \vec{j} \frac{\partial}{\partial y} + \vec{k} \frac{\partial}{\partial z} \right] \phi$$

$$\Rightarrow \vec{f} = \psi \left( \vec{i} \frac{\partial \phi}{\partial x} + \vec{j} \frac{\partial \phi}{\partial y} + \vec{k} \frac{\partial \phi}{\partial z} \right)$$

$$\Rightarrow \vec{f} = \psi \frac{\partial \phi}{\partial x} \vec{i} + \psi \frac{\partial \phi}{\partial y} \vec{j} + \psi \frac{\partial \phi}{\partial z} \vec{k} \quad \text{---} (1)$$

Let $\psi \frac{\partial \phi}{\partial x} = f_1$, $\psi \frac{\partial \phi}{\partial y} = f_2$, $\psi \frac{\partial \phi}{\partial z} = f_3$ — (2)

So (1) becomes

$$\vec{f} = f_1 \vec{i} + f_2 \vec{j} + f_3 \vec{k} \quad \text{---} (3)$$

Now, $\text{curl}\,\vec{f} = \nabla \times \vec{f}$

$$= \left( \vec{i} \frac{\partial}{\partial x} + \vec{j} \frac{\partial}{\partial y} + \vec{k} \frac{\partial}{\partial z} \right) \times \vec{f}$$

$$\therefore \text{curl } \vec{f} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_1 & f_2 & f_3 \end{vmatrix}$$

$$\Rightarrow \text{curl } \vec{f} = \vec{i}\left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}\right) - \vec{j}\left(\frac{\partial f_3}{\partial x} - \frac{\partial f_1}{\partial z}\right)$$

$$+ \vec{k}\left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right) \text{ ------ } (4)$$

~~$\Rightarrow$ curl $\vec{f}$ =~~

Now, $$\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z} = \frac{\partial}{\partial y}\left(\psi \frac{\partial \phi}{\partial z}\right) - \frac{\partial}{\partial z}\left(\psi \frac{\partial \phi}{\partial y}\right)$$

$$= \left[\frac{\partial \psi}{\partial y}\frac{\partial \phi}{\partial z} - \cancel{\psi \frac{\partial^2 \phi}{\partial y \partial z}}\right] - \left(\frac{\partial \psi}{\partial z}\frac{\partial \phi}{\partial y} - \cancel{\psi \frac{\partial^2 \phi}{\partial z \partial y}}\right)$$

$$= \left(\frac{\partial \psi}{\partial y}\frac{\partial \phi}{\partial z} - \frac{\partial \psi}{\partial z}\frac{\partial \phi}{\partial y}\right)$$

~~Similarly~~

$$\therefore \frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z} = \frac{\partial \psi}{\partial y}\frac{\partial \phi}{\partial z} - \frac{\partial \psi}{\partial z}\cdot\frac{\partial \phi}{\partial y}$$

Similarly

$$\frac{\partial f_3}{\partial x} - \frac{\partial f_1}{\partial z} = \frac{\partial \psi}{\partial x}\cdot\frac{\partial \phi}{\partial z} - \frac{\partial \psi}{\partial z}\frac{\partial \phi}{\partial x}$$

$$\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y} = \frac{\partial \psi}{\partial x}\cdot\frac{\partial \phi}{\partial y} - \frac{\partial \psi}{\partial y}\cdot\frac{\partial \phi}{\partial x}$$

$\therefore$ Eq (4) $\Rightarrow$

$$\text{curl } \vec{f} = \vec{i}\left(\frac{\partial \psi}{\partial y}\frac{\partial \phi}{\partial z} - \frac{\partial \psi}{\partial z}\cdot\frac{\partial \phi}{\partial y}\right)$$

$$- \vec{j}\left(\frac{\partial \psi}{\partial x}\frac{\partial \phi}{\partial z} - \frac{\partial \psi}{\partial z}\frac{\partial \phi}{\partial x}\right)$$

$$+ \vec{k}\left(\frac{\partial \psi}{\partial x}\frac{\partial \phi}{\partial y} - \frac{\partial \psi}{\partial y}\frac{\partial \phi}{\partial x}\right) \quad \text{(5)}$$

$\because \vec{f} \cdot \text{curl } \vec{f}$

$$= \left(f_1\vec{i} + f_2\vec{j} + f_3\vec{k}\right)\cdot\left[\vec{i}\left(\frac{\partial \psi}{\partial y}\frac{\partial \phi}{\partial z} - \frac{\partial \psi}{\partial z}\frac{\partial \phi}{\partial y}\right)\right.$$

$$+ \vec{j}\left(\frac{\partial \psi}{\partial z}\frac{\partial \phi}{\partial x} - \frac{\partial \psi}{\partial x}\frac{\partial \phi}{\partial z}\right)$$

$$\left. + \vec{k}\left(\frac{\partial \psi}{\partial x}\frac{\partial \phi}{\partial y} - \frac{\partial \psi}{\partial y}\frac{\partial \phi}{\partial x}\right)\right]$$

$$= f_1\left(\frac{\partial \psi}{\partial y}\frac{\partial \phi}{\partial z} - \frac{\partial \psi}{\partial z}\cdot\frac{\partial \phi}{\partial y}\right)$$

$$+ f_2\left(\frac{\partial \psi}{\partial z}\frac{\partial \phi}{\partial x} - \frac{\partial \psi}{\partial x}\frac{\partial \phi}{\partial z}\right)$$

$$+ f_3\left(\frac{\partial \psi}{\partial x}\frac{\partial \phi}{\partial y} - \frac{\partial \psi}{\partial y}\cdot\frac{\partial \phi}{\partial x}\right)$$

Putting the values of $f_1$, $f_2$ and $f_3$ from eq (2), we have

$\vec{f} \cdot \text{curl}\, \vec{f}$

$$= \psi \frac{\partial \phi}{\partial x}\left[\frac{\partial \psi}{\partial y} \cdot \frac{\partial \phi}{\partial z} - \frac{\partial \psi}{\partial z} \cdot \frac{\partial \phi}{\partial y}\right]$$

$$+ \psi \frac{\partial \phi}{\partial y}\left[\frac{\partial \psi}{\partial z} \cdot \frac{\partial \phi}{\partial x} - \frac{\partial \psi}{\partial x} \cdot \frac{\partial \phi}{\partial z}\right]$$

$$+ \psi \frac{\partial \phi}{\partial z}\left[\frac{\partial \psi}{\partial x} \cdot \frac{\partial \phi}{\partial y} - \frac{\partial \psi}{\partial y} \cdot \frac{\partial \phi}{\partial x}\right]$$

$$= \psi\left[\frac{\partial \psi}{\partial y} \cdot \frac{\partial \phi}{\partial z} \cdot \frac{\partial \phi}{\partial x} - \frac{\partial \psi}{\partial z} \cdot \frac{\partial \phi}{\partial x} \cdot \frac{\partial \phi}{\partial y}\right]$$

$$+ \psi\left[\frac{\partial \psi}{\partial z} \cdot \frac{\partial \phi}{\partial x} \cdot \frac{\partial \phi}{\partial y} - \frac{\partial \psi}{\partial x} \cdot \frac{\partial \phi}{\partial y} \cdot \frac{\partial \phi}{\partial z}\right]$$

$$+ \psi\left[\frac{\partial \psi}{\partial x} \cdot \frac{\partial \phi}{\partial y} \cdot \frac{\partial \phi}{\partial z} - \frac{\partial \psi}{\partial y} \cdot \frac{\partial \phi}{\partial z} \cdot \frac{\partial \phi}{\partial x}\right]$$

$\Rightarrow \vec{f} \cdot \text{curl}\, \vec{f} = 0$

Proved